STUDY ON THE CONSTRUCTION AND APPLICATIONS OF SUPPLEMENTARY DIFFERENCE SETS

A synopsis for the proposed Ph.D. Research work in Mathematics

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1. Title : - Study on the construction and applications of supplementary difference sets

2. State of the art

2.1 Introduction :

In 1972 J. Seberry [1] defined supplementary difference set (SDS) on abelian group, which was a generalization of Difference set and complimentary difference set. Later she also demonstrated application of supplementary difference set in the construction of Hadamard matrix. In 1991 C. Koukouvinous et al. [2] obtained optimal designs with the help of supplementary difference sets. In 1992 Mieko Yamada [3] also introduced relation between supplementary difference set and Jacobi sum.

Followings are the basic terminologies concerning SDS.

- **Difference Set:** Given a group G of order v, a (v, k,λ)-difference set D isa k -subset of G such that the list of differences $\{d_1 - d_2 , such\ that\ d_1 , d_2 \in D\ where\ d_1 \neq d_2)$ contains each non-identity element of G exactly λ times. Formally, for every non-identity element g in G there are exactly λ pairs $(d_1 ,d_2) \in D \times D$ such that$d_1 - d_2$= g.
- **Connection to Hadamard Matrices:** Hadamard matrices are closely related to difference sets through the construction of symmetric designs [4]. If a (v, k, λ)-difference set exists, it can be used to construct a symmetric balanced incomplete block design (BIBD) with parameters (v, k, λ). Specifically, a Hadamard matrix of order 4n can be constructed if there exists a (4n-1, 2n-1, n-1)-difference set.
- **Dzekers Difference Set:** Let G be a group of order 4n-1. A subset D of group G of size 2n-1 is a (4n-1, 2n-1, n-1)-difference set if every non-identity element g in G can be written as $d_i - d_j$ for exactly n-1 pairs d_i-$d_j \in$ D×D.
- **Complementary Difference Set**: The complement of D in G, denoted by D' = G -D is also a difference set. If D s a (4n-1, 2n-1, n-1)-difference set, then D' is a (4n-1, 2n, n)-difference set. This means that every non-identity element g in G can be written as $d_i^{'}$ - $d_j^{'}$ for exactly n pairs($d_i^{'}$-$d_j^{'}$)$\in D^{'} \times D^{'}$
- **Supplementary Difference Set:** Let G be group of order of v and $D_i$be any subset of G where $|D_i| = K_i$, 1≤i≤ n.So, the difference between each element from the eachD_i is equal to a fixed value which give value of λ ,we write this in form {v, k_1,k_2…..k_n,λ } . It also satisfies then,$\lambda(v-1) = \sum_1^n k_i(k_i\text{-}1)$.

- **Cyclotomic Classes:** Cyclotomic classes [3] arise from the division of the multiplicative group of a finite field into equivalence classes based on powers of a primitive rootfor a given integer n, the n-th cyclotomic class C_i modulo q-1 is defined as:

$$C_i = \{ \alpha^{i+kn} \mid K \epsilon Z \ \} \text{ for i =0,1,2...n-1}$$

2.2 Recent Development:

- In 2009 D. Z. Dokovic [4] obtained supplementary difference set with symmetry for Hadamard matrix by the exhaustive computer search.

- In 2016 M. Araya, M. Harada and S. Suda [6] studied skew symmetric 2- {v, r, k, λ} supplementary difference set realated to certain class of complex spherical 2 codes for v≤51.

- In 2020 J. A. Armario and D. L. Flannery [7] introduced almost supplementary difference set for odd m certain ASDS in Z_m that have amicable incidence matrices are equivalent to quaternary sequence of odd length m with optimal autocorrelation .

- In 2024 T. Xia et al. [5] constructed new families of supplementary difference sets, that is 4- $\{q^2;\frac{q^2-1}{8};\frac{q^2-9}{16}\}$ and 4- $\{ q^2;\frac{q(q-1)}{2}$; q(q-2)$\}$.

2.3 Some unresolved cases:

In spite of the diligent search there are numerous parameter sets for which SDS are not constructed some of them are: 4- {59;22,27,28,29;47},4-{65;25,30,31,32;53},4-{67 ; 26,31,32,32;54},4-{77;30,37,37,38;65}.

Skew Hadamard matrices depend upon the (ssss), (ksss), (kkss), (kkks) type SDS defined by D. Z. Dokovic [4]. Following are some parameter sets for which these are not resolved: 4-{63 ; 31,31,29,24;52} of (ksss) type, 4-{63;31,31,29,24;52} of (kkss) type. Also 4-{89;36,41,43,43;74}, 4-{101;41,48,49,49;86}, 4 -{107;44,50,52,52;94} SDS generating skew type Hadamard matrices are still not known [8]. Apart from these cases there are numerous parameter sets for which SDS are not discovered.

3. Problem to be investigated:

In this proposed research work we aim to investigate the following problems:

(i) Enumeration of SDSs for Given Parameters: Determine the exact number of SDSs for given parameters (v, k, λ) and specific groups. This involves counting all possible sets that meet the SDS criteria

(ii) Optimal Construction Methods: P Investigate new and efficient methods for constructing SDSs that lead to Hadamard matrices. Focus on finding constructions that are less complex and more scalable for larger matrices**.**

(iii) SDSs in Non-Abelian Groups: Extend the concept of SDSs to non-abelian groups and investigate their role in the construction of Hadamard matrices.

(iv) Applications Beyond Hadamard Matrices: Explore other combinatorial structures or mathematical objects where SDSs can be applied. Investigate whether these sets have applications in areas like coding theory, cryptography, and design theory.

4. Objective of the research work: Objective of this proposed research works are as follows:

a) Review and Synthesize Existing Knowledge
b) Investigate Unresolved Cases
c) Develop New Construction Methods
d) Generalize SDSs to Non-Abelian Groups
e) Expand Applications Beyond Hadamard Matrices

5. Tools to be used: This research work will depend on the recent development of Supplementary difference set and its application. For this, relevant books and journals are needed. To develop and test the supplementary difference set Exhaustive computer and MATLAB or Python will be used.

6. Contribution to the existing knowledge: Various supplementary difference set have been used to construct Hadamard matrices but, we aim to apply new families of SDS with 4 or more sets. Therefore this research work will surely contribute to existing knowledge.

8. Conclusion: This research work will identify the various new families of SDS but with the help of developed concept and new ideas. We hope to complete this research work and contribute to the society.

Reference:

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